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# Harmonic Univalent Functions for Which Analytic Part is Starlike (Study on Non-Analytic and Univalent Functions and Applications)

AUTHOR(S):

POLATOGLU, Yasar

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CITATION:

POLATOGLU, Yasar. Harmonic Univalent Functions for Which Analytic Part is Starlike (Study on Non-Analytic and Univalent Functions and Applications). 数理解析研究所講究録 2009, 1626: 93-98

ISSUE DATE:

2009-01

URL:

<http://hdl.handle.net/2433/140313>

RIGHT:

# Harmonic Univalent Functions for Which Analytic Part is Starlike

Yaşar POLATOĞLU

## Abstract

In a simply connected domain  $\mathcal{U} \subset \mathbb{C}$  a complex-valued harmonic function  $f$  has the representation  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic function in  $\mathcal{U}$ , and are called analytic part and co-analytic part of  $f$ , respectively.

Let  $h(z) = a_0 + a_1z + a_2z^2 + \cdots$  and  $g(z) = b_0 + b_1z + b_2z^2 + \cdots$  be analytic functions in the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ , if  $(|h'(z)|^2 - |g'(z)|^2) > 0$  or  $(|h'(z)|^2 - |g'(z)|^2) < 0$  then  $f$  is called a sense-preserving harmonic function in  $\mathbb{D}$ . The class of all sense-preserving harmonic functions in  $\mathbb{D}$  with  $a_0 = b_0 = 0$ , and  $a_1 = 1$  will be denote by  $\mathcal{S}_H$ . Thus  $\mathcal{S}_H$  contains the standard class of analytic univalent functions  $\mathcal{S}$ .

The aim of this paper is to investigate the subclass of  $\mathcal{S}_H$ . By choosing only these functions whose analytic parts are starlike functions for such mappings we will find distortion theorems.

## 1 Introduction

Let  $h(z) = z + a_2z^2 + \cdots$  be an analytic function in the open unit disc  $\mathbb{D}$ , if  $h(z)$  satisfies the condition

$$\operatorname{Re} \left( z \frac{h'(z)}{h(z)} \right) > 0, \quad (1.1)$$

then  $h(z)$  is called starlike function in  $\mathbb{D}$ , and the class of starlike functions in  $\mathbb{D}$  is denoted by  $\mathcal{S}^*$  [3].

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2000 *Mathematics Subject Classification*: Primary 30C45.

*Key words and phrases*: harmonic univalent functions, starlike functions, sense-preserving.

Next  $\Omega$  be the family of functions  $\phi(z)$  which are regular and satisfying the conditions  $\phi(0) = 0$ ,  $|\phi(z)| < 1$  for every  $z \in \mathbb{D}$ , and let  $\Omega(a)$  be the class of functions  $w(z)$  which are analytic in  $\mathbb{D}$  and satisfying the conditions  $w(0) = a$ ,  $|w(z)| < 1$  for all  $z \in \mathbb{D}$ . We note that  $\Omega_{\mathcal{H}}(a)$  be the union of all classes  $\Omega(a)$  whereas  $a$  ranges over  $[0, 1)$  [4].

Moreover, a function  $f$  is said to be a complex-valued harmonic function in  $\mathbb{D}$  if both  $\operatorname{Re} f$  and  $\operatorname{Im} f$  are real harmonic in  $\mathbb{D}$ . Every such  $f$  can be uniquely represented by

$$f = h + \bar{g}, \quad (1.2)$$

where  $h$  and  $g$  are analytic in  $\mathbb{D}$  with  $g(0) = 0$ . A complex valued harmonic function  $f$  not identically constant, satisfying (1.2) is said to be sense-preserving in  $\mathbb{D}$  if and only if satisfying the equation

$$g'(z) = w(z)h'(z), \quad (1.3)$$

where  $w(z)$  is analytic in  $\mathbb{D}$  with  $|w(z)| < 1$  for every  $z \in \mathbb{D}$  [2], and the function  $w(z)$  is called the second dilatation of  $f$ . It is closely related to the Jacobian of  $f$  is defined by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2. \quad (1.4)$$

Finally, let  $h(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=2}^{\infty} b_n z^n$  be analytic functions in  $\mathbb{D}$ . Choose  $g(0) = 0$ , i.e.,  $b_0 = 0$ , so the representation (1.2) is unique in  $\mathbb{D}$  and is called the canonical representation of  $f$ . For univalent and sense-preserving harmonic functions  $f$  in  $\mathbb{D}$ , it is convenient to make further normalization (with no loss generality)  $h(0) = 0$ , i.e.,  $a_0 = 0$  and  $h'(0) = 1$ , i.e.,  $a_1 = 1$ . The family of all such functions  $f$  is denoted by  $\mathcal{S}_{\mathcal{H}}$  [1], [2], [5].

In this paper we will study on the subclass of  $\mathcal{S}_{\mathcal{H}}$  consisting of all univalent harmonic functions for which analytic part is starlike, and this class will be denoted by  $\mathcal{S}_{\mathcal{H}}^*$ .

## 2 Main Results

**Lemma 2.1.** *Let  $w(z)$  be element of  $\Omega(a)$ , then*

$$\frac{|a - r|}{1 - ar} \leq |w(z)| \leq \frac{a + r}{1 + ar} \quad (2.1)$$

for all  $z \in \mathbb{D}$ , and  $0 \leq a < 1$ .

*Proof.* Let  $w(z) \in \Omega(a)$ , then  $w(z)$  is analytic in  $\mathbb{D}$  and satisfies the condition  $w(0) = a$ . Now we consider the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, \quad z \in \mathbb{D}. \quad (2.2)$$

Therefore  $\phi(z)$  satisfies the conditions of Schwarz lemma. Using the estimate the Schwarz lemma  $|\phi(z)| \leq |z| = r$ , which gives

$$|\phi(z)| = \left| \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} \right| \leq r. \quad (2.3)$$

The inequality (2.3) can be written in the following form

$$\left| \frac{w(z) - a}{1 - \overline{a}w(z)} \right| \leq r \Leftrightarrow |w(z) - a| \leq r|aw(z) - 1|. \quad (2.4)$$

The inequality (2.4) is equivalent

$$\left| w(z) - \frac{a(1 - r^2)}{1 - a^2 r^2} \right| \leq \frac{r(1 - a^2)}{1 - a^2 r^2}. \quad (2.5)$$

The equality holds in the inequality (2.5) only for the function

$$w(z) = e^{i\beta} \frac{e^{i\theta} z + a}{1 + \overline{a}e^{i\theta} z}, \quad z \in \mathbb{D}.$$

From the inequality (2.5) we have

$$\begin{aligned} |w(z)| &\geq \left| \frac{a(1 - r^2)}{1 - a^2 r^2} - \frac{r(1 - a^2)}{1 - a^2 r^2} \right| = \frac{|a - r|}{1 - ar}, \\ |w(z)| &\leq \frac{a(1 - r^2)}{1 - a^2 r^2} + \frac{r(1 - a^2)}{1 - a^2 r^2} = \frac{a + r}{1 - ar}. \end{aligned} \quad (2.6)$$

□

**Corollary 2.2.** Let  $f \in \mathcal{S}_{\mathcal{H}}^*$ , then

$$\frac{(1 - r)|a - r|}{(1 + r)^3(1 - ar)} \leq |g'(z)| \leq \frac{(1 + r)(a + r)}{(1 - r)^3(1 + ar)}. \quad (2.7)$$

*Proof.* Let  $f \in \mathcal{S}_{\mathcal{H}}^*$ , then

$$g'(z) = w(z)h'(z) \Leftrightarrow \frac{g'(z)}{h'(z)} = w(z). \quad (2.8)$$

Using lemma 2.1 we obtain

$$|h'(z)| \frac{|a-r|}{(1-ar)} \leq |g'(z)| \leq |h'(z)| \frac{a+r}{(1+ar)}. \quad (2.9)$$

On the other hand, since  $h(z)$  is starlike, then  $h(z)$  satisfies the following inequality

$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}. \quad (2.10)$$

Considering the inequalities (2.9) and (2.10) together we obtain (2.7).  $\square$

**Theorem 2.3.** Let  $f \in \mathcal{S}_{\mathcal{H}}^*$ , then

$$|g(z)| \leq \log \left( \frac{1+ar}{1-r} \right)^{\left(\frac{a-1}{a+1}\right)^2} + \frac{a-3}{a+1} \cdot \frac{r}{1-r} + \frac{2r^2}{(1-r)^2}. \quad (2.11)$$

*Proof.* Applying the estimate (2.7) we have

$$|g(z)| = \left| \int_C g'(\zeta) d\zeta \right| \leq \int_C |g'(\zeta)| |d\zeta| \leq \int_0^r \frac{(1+\rho)(a+\rho)}{(1-\rho)^3(1+a\rho)} d\rho.$$

Integrating, we obtain the estimate (2.11), where  $C = [0, z]$  is Jordan arc.  $\square$

**Theorem 2.4.** Let  $f$  be element of  $\mathcal{S}_{\mathcal{H}}^*$ , then

$$\begin{aligned} & \log \left( \frac{1+r}{1+ar} \right)^{\left(\frac{1+a}{1-a}\right)^2} - \frac{2r[(1+a)r+2a]}{(1-a)(1+r)^2} \leq |f(z)| \\ & \leq \frac{r(1+2r)}{(1-r)^2} + \frac{a-3}{a+1} \cdot \frac{r}{1-r} + \log \left( \frac{1+ar}{1-r} \right)^{\left(\frac{a-1}{a+1}\right)^2}. \end{aligned}$$

*Proof.* Let  $z \in \mathbb{D}$ . We denote  $|z| = r$  and  $M(r) = \inf\{|f(z)| : |z| = r\}$ . Then  $|f(z)| \geq M(r)$  and  $\{w : |w| \leq M(r)\} \subset f(\{\zeta : |\zeta| < r\}) \subset f(\mathbb{D})$ , hence there exist  $z_r$  satisfying  $|z_r| = r$  such that  $M(r) = |f(z_r)|$  and  $\gamma(t) = tf(z_r)$ ,

$t \in [0, 1]$ . Therefore  $f^{-1}(\gamma(t)) = \Gamma(t)$ ,  $t \in [0, 1]$  is well defined Jordan arc, and

$$\begin{aligned} |f(z)| &\geq \int_{f^{-1}(\gamma(t))} (|h'(\zeta)| - |g'(z)|) |d\zeta| = \int_{f^{-1}(\gamma(t))} (|h'(\zeta)| - |w(\zeta)||h'(z)|) |d\zeta| \\ &= \int_{f^{-1}(\gamma(t))} |h'(\zeta)|(1 - |w'(z)|) |d\zeta| \geq \int_0^r \left( \frac{1-\rho}{(1+\rho)^3} \right) \left( 1 - \frac{a+\rho}{1+a\rho} \right) d\zeta \Rightarrow \\ |f(\zeta)| &\geq \log \left( \frac{1+r}{1+ar} \right)^{\left( \frac{1+a}{1-a} \right)^2} - \frac{2r[(1+a)r+2a]}{(1-a)(1+r)^2}. \end{aligned}$$

On the other hand, we have

$$|f(z)| = |h(z) + \overline{g(z)}| \leq |h(z)| + |g(z)|,$$

then using the following inequality and theorem 2.3, we have

$$|f(z)| \leq \frac{r(1+2r)}{(1-r)^2} + \frac{a-3}{a+1} \cdot \frac{r}{1-r} + \log \left( \frac{1+ar}{1-r} \right)^{\left( \frac{a-1}{a+1} \right)^2}.$$

□

**Theorem 2.5.** Let  $f \in \mathcal{S}_{\mathcal{H}}^*$ , then

$$\frac{(1-a^2)(1-r)}{(1+ar)^2(1+r)^5} \leq J_f(z) \leq \frac{(1-ar)^2 - |a-r|^2}{(1-ar)^2(1-r)^6}.$$

*Proof.* Using lemma 2.1 and the following relations,

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2,$$

$$g'(z) = w(z)h'(z),$$

and after the simple calculations we get

$$\frac{(1-a^2)(1-r)}{(1+ar)^2(1+r)^5} \leq |h'(z)|^2(1 - |w(z)|^2) \leq \frac{(1-ar)^2 - |a-r|^2}{(1-ar)^2(1-r)^6}.$$

□

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YAŞAR POLATOĞLU

Department of Mathematics and Computer Science,  
İstanbul Kültür University, 34156 İstanbul, Turkey  
e-mail: y.polatoglu@iku.edu.tr